

Quadratic Elements

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September 23, 2016

Motivated by question 0.x.4 in the text, define mesh:

$$0 = x_0 < x_{\frac{1}{2}} < x_1 < x_{\frac{3}{2}} < x_2 < \cdots < x_{n-\frac{1}{2}} < x_n = 1,$$

and grid spacing, $h_j = x_j - x_{j-1}$, $j = 1, \dots, n$. Claim: basis functions associated with mesh nodes satisfy

$$\phi_j(x) = \begin{cases} \frac{2}{h_j^2}(x - x_{j-1})(x - x_{j-1/2}), & \text{if } x_{j-1} \leq x < x_j \\ \frac{2}{h_{j+1}^2}(x_{j+1} - x)(x_{j+1/2} - x), & \text{if } x_j \leq x < x_{j+1}, \quad j = 1, 2, \dots, n, \\ 0, & \text{otherwise} \end{cases}$$

and those associated with the midpoints are

$$\phi_{j-\frac{1}{2}}(x) = \begin{cases} \frac{4}{h_j^2}(x_j - x)(x - x_{j-1}), & \text{if } x_{j-1} \leq x < x_j \\ 0, & \text{otherwise} \end{cases} \quad j = 1, 2, \dots, n.$$

Tasks:

1. Discuss basis functions. Reasonable? Why?
2. Calculate stiffness matrix for these elements
3. Solve our canonical problem,

$$-u''(x) = \sin 2\pi x, \quad u(0) = 0, \quad u'(1) = 0$$

4. Numerical convergence study. Matches analysis?